INVARIANT FIELDS OF SYMPLECTIC AND ORTHOGONAL GROUPS

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Abstract The projective orthogonal and symplectic groups $PO_n(F)$ and $PSp_n(F)$ have a natural action on the F vector space $V' = M_n(F) \oplus \ldots \oplus M_n(F)$. Here we assume F is an infinite field of characteristic not 2. If we assume there is more than one summand in V', then the invariant fields $F(V')^{PO_n}$ and $F(V')^{PSp_n}$ are natural objects. They are, for example, the centers of generic algebras with the appropriate kind of involution. This paper considers the rationality properties of these fields, in the case $1, 2$ or 4 are the highest powers of 2 that divide n. We derive rationality when n is odd, or when 2 is the highest power, and stable rationality when 4 is the highest power. In a companion paper [ST] joint with Tignol, we prove retract rationality when 8 is the highest power of 2 dividing n. Back in this paper, along the way, we consider two generic ways of forcing a Brauer class to be in the image of restriction.

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Introduction

In this paper F will always be an infinite field of characteristic not 2. Set $V = M_n(F) \oplus \ldots \oplus M_n(F)$ to be a sum of $r \geq 2$ copies of $n \times n$ matrices. This V is a representation of the projective linear group $PGL_n = GL(F)/F^*$ induced by conjugation acting diagonally. The invariant field $F(V)^{PGL_n}$ has become an algebraic and geometric object of remarkable importance, partly because $F(V)^{PGL_n}$ is the center $Z(F,n,r)$ of the so called generic division algebra $UD(F,n,r)$. Procesi observed that $Z(F,n,r)$ is rational (i.e. purely transcendental) over $Z(F,n,2)$. For this reason, it makes sense to only consider the case $V = M_n(F) \oplus M_n(F)$.

In addition, there are important subgroups $\mathcal{G}$ of PGL_n where the invariant field $F(V)^{\mathcal{G}}$ is also of significance. In particular, we will be interested in $\mathcal{G} = PO_n$ the projective orthogonal group, and when n is even, $\mathcal{G} = PSp_n$ the projective symplectic group. Note that these groups will be defined in detail in section one. By [P2] p. 377-378 and [R] p. 184, $F(V)^{\mathcal{G}}$ is the center of the generic central simple algebra with orthogonal respectively symplectic involution.

There is a rich theory of central simple algebras with involution, which make it immediately clear that $F(V)^{\mathcal{G}}/F$ is, for both the $\mathcal{G}$ above, retract rational over F when m is odd and $n = m$, $2m$, or $4m$. In a companion paper [ST], joint with J.-P. Tignol, we will prove retract rationality in the case $n = 8m$. In the case $n = 2^r m$ for $r > 3$, retract rationality is equivalent to questions about central simple algebras that, as of now, have no answer. But in the $r = m$, $2m$, $4m$ cases, one can ask whether $F(V)^{\mathcal{G}}$ has stronger properties, for example, that this field is rational or stably rational over F. Recall that K/F is stably rational if and only if there is a field $L \supset K$ such that L/F and L/K are rational.

In the case $n = 4m$, we show in section five that $F(V)^{\mathcal{G}}/F$ is stably rational, a not surprising result. It is somewhat more difficult to see that $F(V)^{\mathcal{G}}/F$ is rational in the cases $n = m$ and particularly when $n = 2m$. The main part of this paper is concerned with proving this last fact. Along the way we introduce 2 "generic" ways of forcing a Brauer class to be in the image of restriction.

For the reader's convenience, let us summarize the known results in:

Theorem. *Let F be an infinite field of characteristic not 2. Let $V = M_n(F) \oplus M_n(F)$ with the standard action of PGL_n. View $PO_n, PSp_n \subset PGL_n$. Then*

a) If n is odd, $F(V)^{PO_n}/F$ is rational.

b) If $n = 2m$ and m is odd, then $F(V)^{PSp_n}/F$ and $F(V)^{PO_n}/F$ are both rational.

c) If $n = 4m$ and m is odd, then $F(V)^{PSp_n}/F$ and $F(V)^{PO_n}/F$ are both stably rational.

d) If $n = 8m$ and m is odd, then $F(V)^{PSp_n}/F$ and $F(V)^{PO_n}/F$ are retract rational.

Note that part a) is 1.2 below, part b) is 4.1 and 4.2 below, and part c) is 5.1 below. Part d) is proved in the above mentioned paper [ST], joint work with J.-P. Tignol. Let us, for completeness sake, record the general version of the above result (with a sketch of a proof). Recall that for an algebraic group $\mathcal{G}$, a **good** representation (always algebraic representation) is one where a generic point has trivial stabilizer.

Corollary. *Let $\mathcal{G} = PO_n$ for all n or PSp_n for n even. Let m be an odd integer.*

a) Let $V' = M_n(F) \oplus \ldots \oplus M_n(F)$ be a direct sum of at least two copies of $M_n(F)$ with the usual action by $\mathcal{G}$. If $n = m$ or $2m$, then $F(V')^{\mathcal{G}}$ is rational over F. If $n = 4m$, $F(V')^{\mathcal{G}}$ is stably rational over F. If $n = 8m$, then $F(V')^{\mathcal{G}}$ is retract rational over F.

b) Let V'' be a good representation of $\mathcal{G}$ and F algebraically closed of characteristic 0. If $n = m$, $2m$, or $4m$, then $F(V'')^{\mathcal{G}}$ is stably rational over F. If $n = 8m$, then $F(V'')^{\mathcal{G}}$ is retract rational over F.

Proof. Of course, we have not defined retract rational here. We refer the reader to [S] or [LN] p. 77 for the definition and basic properties. To prove part a), note that a proof parallel to that of A.3 shows that $F(V')^{\mathcal{G}}/F(V)^{\mathcal{G}}$ is rational. (Alternatively, $F(V')^{\mathcal{G}}$ is the center of the generic central simple algebra in the appropriate number of variables and with the appropriate involution. This is known to be rational over the same object with 2 variables, i.e., $F(V)^{\mathcal{G}}$.) In part b), the assumption on F is only necessary to apply Bogomolov's no name lemma, so $F(V'')^{\mathcal{G}}$ is stably isomorphic to $F(V)^{\mathcal{G}}$. This proves b). ■

To begin the paper proper, let us define some notation and recall some constructions. F will always be our ground field. If $\mathcal{G}$ is an algebraic group, an F representation of $\mathcal{G}$ is a finite dimensional F vector space V and a map $\mathcal{G} \to GL_F(V)$ which is a homomorphism as a map between algebraic groups. If $\phi : K \to L$ is an embedding of fields, and V is a K vector space, we write $V \otimes_\phi L$ to mean the tensor product $V \otimes_K L$ where L is a K vector space via ϕ. If A/K, B/K' are algebras over K, K' respectively, and $K' \subset K$, we abbreviate $A \otimes_K (B \otimes_{K'} K)$ as $A \otimes_K B$.

Suppose K, L are fields regular and separably generated over F. Then $K \otimes_F L$ is a domain. The field of fractions, $q(K \otimes_F L)$, of $K \otimes_F L$ we call the join of K, L over F. Suppose next that K/F is a finite separable extension, and L/K is separably generated. Let $K' \supset K \supset F$ be the Galois closure of K/F, and let $H = \mathrm{Gal}(K'/K)$, $G = \mathrm{Gal}(K'/F)$. Choose coset representatives σ_i, of H in G where $i = 1, \ldots, r$. That is, assume $G = \cup_{i=1}^r \sigma_i H$ is a disjoint union. We can also view σ_i as an embedding of K in K', and we write $\sigma_i(L) = L \otimes_{\sigma_i} K'$. Since we may assume $\sigma_1 = 1$, $\sigma_1(L) = K' \otimes_K L$ has an action by H via the action on K'. If we set T to be the field join $L\sigma_2(L) \ldots \sigma_r(L)$ over K', then T has an natural G action extending that on K'. The fixed field T^G will be written $\mathrm{Tr}_{K/F}(L)$ and is called the transfer. Note that if L is the field of rational functions of a K variety V, then $\mathrm{Tr}_{K/F}(L)$ is the field of rational functions of the transfer variety of V to F. Also note that if L/K is rational, then $\mathrm{Tr}_{K/F}(L)/F$ is rational.

Suppose G is a finite group and L is a field with a (perhaps trivial) G action. Let M be a lattice over G. That is, let M be a finite generated $\mathbb{Z}[G]$ module which is free as an abelian group. The group ring is, as a ring, just the Laurent polynomial ring $L[x_1, x_1^{-1}, \ldots, x_r, x_r^{-1}]$ and so $L[M]$ is a domain. The action of G on L and M induces an action on $L[M]$, which in turn induces an action on the field of fractions $q(L[M]) = L(M)$.

We will also require twistings of the above construction. Suppose $\alpha \in \mathrm{Ext}(M, L^*)$. That is, suppose α is an extension $0 \to L^* \to M' \to M \to 0$. Then there is an induced action of G on $L[M]$ such that $L[M]^* \cong M'$. We write $L_\alpha[M]$ to mean $L[M]$ with this

α twisted action. We write $L_\alpha(M)$ to mean $q(L_\alpha[M])$ with the extended action. In this paper we will make significant use of group cohomology, to which we refer the reader to [B] as a good general reference.

Finally, we recall a bit about central simple algebras and the Severi-Brauer variety. Let A/K be a central simple algebra by which we mean that A is finite dimensional over its center K and simple. The dimension of A/K is always a square, n^2, and we call n the degree of A. Of course such an A always has the form of matrices $M_r(D)$ over a division algebra D/K. We say A/K, B/K are Brauer equivalent if they are matrices over the same division algebra, or equivalently if $M_r(A) \cong M_s(B)$ for some r, s. The equivalence classes under this relationship form, of course, the Brauer group of K which we write as $\mathrm{Br}(K)$. Recall that the product is induced by tensor product and the inverse of the class of A/K is the class of the opposite algebra A°. If $L \supset K$, then $A \to A \otimes_K L$ induces a group homomorphism $\mathrm{Br}(K) \to \mathrm{Br}(L)$ we call restriction. We also use the word restriction to refer to the map $A \to A \otimes_K L$ on algebras (and not just classes).

If L/K is G Galois, and $\gamma \in H^2(G, L^*)$, then we can form the crossed product $\Delta(L/K, G, \gamma)$. By this we mean the algebra $\oplus_{g \in G} L u_g$ where $u_g x = \sigma(x) u_g$ for all $x \in L$, and $u_g u_h = c(g,h) u_{gh}$ for a 2 cocycle $c(g,h)$ in γ. The crossed product induces an isomorphism $H^2(G, L^*) \to \mathrm{Br}(L/K)$, where $\mathrm{Br}(L/K)$ is the kernel of the restriction map $\mathrm{Br}(K) \to \mathrm{Br}(L)$. Some of the properties of central simple algebras which we need are contained in the classical:

Theorem 0.1. *(e.g. [J] p. 226f, [LN] p.34) a) Suppose A/K, B/K are central simple algebras of degree n and are also Brauer equivalent. Then $A \cong B$ over K.*

b) Every Brauer class contains a unique division algebra, which is the member of the class of minimal degree.

c) Suppose A/K has degree n, $r \geq 1$ and (r, n) is the gcd. Then the Brauer class of $A \otimes_K \ldots \otimes_K A$ (r times) contains an element of degree dividing $n/(r, n)$.

In particular in any Brauer equivalence class we can talk about **the** central simple algebra of degree n, assuming such exists.

Let us recall and record the useful result of Endo-Miyata ([EM] or e.g. [LN] p. 82).

Proposition 0.2. *Let G be a finite group. Let L/K be a G Galois extension and V an L vector space with a semilinear G action. Then $L(V)^G/K$ is a rational extension. As a consequence, if P is a permutation G lattice, then $L(P)^G/K$ is rational. As another consequence, if V, W are faithful F representations of G, then $F(V \oplus W)^G/F(V)^G$ is rational. In particular, $F(V)^G$ and $F(W)^G$ are stably isomorphic.*

Suppose A/K is central simple of degree n. The Severi-Brauer variety $SB(A)$ (e.g. [LN] p.89 for what follows) can be defined as the variety of dimension n right ideals of A, realized as a closed subvariety of the Grassmann variety of subspaces of A of dimension n. If $A = \mathrm{End}_K(V)$, then $SB(A)$ is isomorphic to the projective space $\mathbb{P}_K(V)$. It follows that $SB(A)$ is irreducible and has a field of fractions we write as $K(A)$. Note that $K(A)$ is the generic splitting field of A introduced by Amitsur. In particular, $A \otimes_K K(A) \cong M_n(K(A))$. If A is split, $SB(A)$ is projective space so $K(A)$

is rational over K. As a matter of notation, if A/K' is central simple and $K' \subset K$, we will abbreviate $K(A \otimes_{K'} K)$ as $K(A)$. We quote two results about the birational isomorphisms of Severi-Brauer varieties.

Theorem 0.3. *Suppose A/K is central simple of degree n.*

a) (Amitsur e.g. [LN] 13.29): $K(A) \cong K(A^\circ)$.

b) (Tregub [T]): Suppose n is odd. Then $K(A) \cong K(B)$, where B is the element of the Brauer class of $A \otimes_K A$ of degree n.

As a tool in what follows, we need a different description of $K(A)$. To begin this discussion, let $G \supset H$ be a finite group and subgroup, and K/F a separable $G - H$ extension. That is, if L/F is the Galois closure of K/F, then G is the Galois group of L/F and H is the subgroup corresponding to L/K. Form the G lattice $\mathbb{Z}[G/H]$. Recall that this is the lattice with $\mathbb{Z}$ basis $\{u_{gH} | gH \in G/H\}$ such that $g'(u_{gH}) = u_{g'gH}$. In $\mathbb{Z}[G/H]$ form the sublattice $I \subset \mathbb{Z}[G/H]$ generated by all $u_{gH} - u_{g'H}$. Note that there is an exact sequence $0 \to I \to \mathbb{Z}[G/H] \to \mathbb{Z} \to 0$. The boundary map $\delta : \mathbb{Z} = H^0(G, \mathbb{Z}) \to H^1(G, I)$ defines a 1 cohomology class $\alpha \in H^1(G, I)$ we call the canonical class. One can compute that α has order $[G : H]$, generates $H^1(G, I)$, and the restriction $\alpha|_H = 0$.

Lemma 0.4. *a) Let M be any G module. There is a map $d : \mathrm{Ext}(I, M) \to H^2(G, M)$, natural in M, whose image is precisely those element of $H^2(G, M)$ that split when restricted to H. This map is one to one if $H^1(H, M) = 0$.*

b) If $\beta : 0 \to M \to M' \to I \to 0$ corresponds to $\beta \in \mathrm{Ext}(I, M)$, then $d(\beta) = \delta(\alpha)$ where δ is the boundary in the long exact sequence associated to β.

Proof. There is an exact sequence

$$\mathrm{Ext}^1(\mathbb{Z}[G/H], M) \to \mathrm{Ext}^1(I, M) \to \mathrm{Ext}^2(Z, M) \to \mathrm{Ext}^2(\mathbb{Z}[G/H], M).$$

Now $\mathrm{Ext}^i(\mathbb{Z}[G/H], M) = H^i(G, \mathrm{Hom}(\mathbb{Z}[G/H], M)) =$ (by Shapiro's Lemma e.g. [B] p.73) $= H^i(H, M)$. Similarly, $\mathrm{Ext}^i(\mathbb{Z}, M) = H^i(G, \mathrm{Hom}(\mathbb{Z}, M)) = H^i(G, M)$. Thus the above exact sequence reads

$$0 \to H^1(H, M) \to \mathrm{Ext}^1(I, M) \to H^2(G, M) \to H^2(H, M).$$

The map $H^2(G, M) \to H^2(H, M)$ is easily seen to be restriction, proving a). Part b) is an easy computation. ∎

The above lemma is sort of a "splitting module" construction. Assume $H^1(H, M) = 0$ and $\gamma \in H^2(G, M)$ is split when restricted to H. Then γ defines a unique $\beta \in \mathrm{Ext}(I, M)$ with $d(\beta) = \gamma$. Let $0 \to M \to M' \to I \to 0$ correspond to β. Since β obviously splits in M', γ splits in M'. Suppose $f : M \to N$ is any morphism of G modules with $H^1(H, N) = 0$. Then clearly f extends to M' if and only if $f_*(\gamma) = 0$.

The above discussion makes it natural to apply 0.4 to the case $M = L^*$, where L is a field with faithful G action. Of course, $H^1(H, L^*) = 0$ by Hilbert's theorem 90. If $K = L^H$ and $F = L^G$, then the set of $\gamma \in H^2(G, L^*)$ that restrict to 0 on H is precisely the relative Brauer group $\mathrm{Br}(K/F)$. If $\gamma \in \mathrm{Br}(K/F)$, let A/F be the corresponding central simple algebra with maximal subfield K and $\beta : 0 \to L^* \to M' \to I \to 0$ the corresponding extension. Let $L_\beta(I)$ be as defined above.

Theorem 0.5. *$L_\beta(I)^G$ is the Amitsur generic splitting field of A. That is, $L_\beta(I)$ is the function field of the Severi-Brauer variety of A.*

Remark. This is well known. When $H = 1$ a proof appears in [LN] p. 95 except that there is an exponent error changing A to A°. This error turns out to be unimportant because of Amitsur's result 0.1 a). We sketch a proof because a precise reference, in this generality, and in anything like this language, is hard to come by.

Proof. As is well known, $K \otimes_F A^\circ \cong \mathrm{End}_K(A)$ where A is a K vector space by the left action and A° acts on A by right multiplication. Thus $L \otimes_F A^\circ \cong \mathrm{End}_L(L \otimes_K A)$. The G action on $L \otimes_F A^\circ$ induces a G action on $\mathbb{P}_L(L \otimes_K A)$ and the Severi-Brauer variety of A° is the variety given by descent.

Viewing $A^\circ \subset \mathrm{End}_L(L \otimes_K A) \subset \mathrm{End}_F(L \otimes_K A)$, let B be the centralizer of A° in $\mathrm{End}_F(L \otimes_K A)$. Since the centralizer of L in B is the centralizer of $L \otimes_K A^\circ$ in $\mathrm{End}_F(L \otimes_K A)$, this centralizer is L. Thus B is Brauer equivalent to A with maximal subfield L. That is, $B = \Delta(L/F, G, \gamma)$. We can choose $u_g \in B$ such that $u_g x = g(x) u_g$ for all $x \in L$. Any such u_g must form a basis of B over L and $u_g u_{g'} = c(g, g') u_{gg'}$ where $c(g, g')$ is a cocycle in γ (see for example the proof of 7.2 in [LN]). If $\beta \in B$ and $w \in L \otimes_K A$ we write $\beta \cdot w$ for the action of β on w. If $g \in G$ and $L\alpha$ is a line in $L \otimes_K A$, i.e. a point in $\mathbb{P}_L(L \otimes_K A)$, then it is easy to check that $g(L\alpha) = u_g \cdot L\alpha$ and this, of course, is independent of the choice of u_g (see [LN] p. 94).

We can choose the u_h for $h \in H$ such that $u_h \cdot (x \otimes a) = h(x) \otimes a$ for all $x \otimes a \in L \otimes_K A$. Then for any coset $gH \subset G$, we can write v_{gH} as the element $u_g \cdot (1 \otimes 1)$. Since the u_g span B, it is clear that the v_{gH} are an L basis of $L \otimes_K A$ and we use this basis to define projective coordinates for $\mathbb{P}_L(L \otimes_K A)$. Specifically, let the line $L(\sum_{gH} a_{gH} v_{gH})$ have projective coordinates $(a_{gH})_{gH \in G/H}$. Note that

$$u_{g'} \cdot L(\sum_{gH} a_{gH} v_{gH}) = L(\sum_{gH} g'(a_{gH}) c(g', g) v_{g'gH}).$$

Define the rational functions d_{gH} for $gH \neq H$ by $d_{gH}(L(\sum_{gH} a_{gH} v_{gH})) = a_{gH}/a_H$, so the d_{gH} are a transcendence basis for $L(\mathbb{P}_L(L \otimes_K A))$. Finally,

$$(g'(d_{g''H}))(L(\sum_{gH} a_{gH} v_{gH})) = g'(d_{g''H}(u_{g'^{-1}} \cdot L(\sum_{gH} a_{gH} v_{gH}))) =$$

$$g'(d_{g''H}(L(\sum_{gH} g'^{-1}(a_{gH}) c(g'^{-1}, g) v_{g'^{-1}gH}))) =$$

Using the definition of $d_{g''H}$ this is:

$$(a_{g'g''H}/a_{g'H}) g'(c(g'^{-1}, g'g'')/c(g'^{-1}, g')) = (a_{g'g''H}/a_{g'H}) c(g', g'')^{-1}.$$

Thus $g'(d_{g''H}) = (d_{g'g''H}/d_{g'H}) c(g', g'')^{-1}$. Replacing A° by A and $c(g', g'')^{-1}$ by $c(g', g'')$ we have the result. ∎

It is convenient to use 0.5 to reprove the following result.

Corollary 0.6. *Let $Z = Z(F,n)$ be the center of the generic division algebra $UD = UD(F,n)$. The generic splitting field $Z(UD)$ is rational over F.*

Proof. We recall that $Z(F,n) = F(M)^{S_n}$ where there is an exact sequence $0 \to M \to M' \to I[S_n/S_{n-1}] \to 0$ and M' is a permutation lattice with $M' = M'' \oplus \mathbb{Z}[S_n/S_{n-1}]$. The cocycle $\gamma \in H^2(S_n, M)$ defines the Brauer group element given by UD. By 0.5, $Z(UD) = F(M')^{S_n}$. But $F(\mathbb{Z}[S_n/S_{n-1}])^{S_n}$ is rational over F since $F(\mathbb{Z}[S_n/S_{n-1}]) = F(x_1, \ldots, x_n)$ with the usual action. By Endo-Miyata (0.2), $F(M')^{S_n}/F(\mathbb{Z}[S_n/S_{n-1}])^{S_n}$ is rational. ∎

We need to make an observation similar to 0.4, but involving cohomology of degree one higher. Let $H \subset G$ and I be as in 0.4. For convenience, assume $G = H \cup HgH$. It immediately follows that $u_H - u_{gH}$ generates I. Choose $H' \subset H \cap gHg^{-1}$. Then there is surjection $\mathbb{Z}[G/H'] \to I$ defined by sending the canonical H' generator to $u_H - u_{gH}$. Define J to be the kernel of this map. Let M be a G module. Assume $\beta \in H^2(H, M)$. Denote by $g(\beta)$ the induced element of $H^2(gHg^{-1}, M)$, and $\beta - g(\beta)$ the obvious induced element of $H^2(H \cap gHg^{-1}, M)$.

Proposition 0.7. *Let M be a G module such that $H^1(H', M) = 0$.*

a) Let $A \subset H^2(H, M)$ be the subgroup consisting of those β such that the restriction of $\beta - g(\beta)$ to H' is 0. For each $\beta \in A$, there is an extension $0 \to M \to M' \to J \to 0$, corresponding to $\alpha \in \mathrm{Ext}(J, M)$ such that $\alpha = 0$ if and only if β is in the image of $H^2(G, M)$. This correspondence is functorial in M.

b) Suppose further that $H^3(G, M) \to H^3(H, M)$ is injective. Then this correspondence is onto $\mathrm{Ext}(J, M)$.

c) Suppose $0 \to M \to M' \to J \to 0$ corresponds to $\beta \in H^2(H, M)/\operatorname{Res}(H^2(G, M))$. Then the image of β generates the kernel of

$$H^2(H, M)/\operatorname{Res}(H^2(G, M) + \delta(H^1(H, J))) \to H^2(H, M')/\operatorname{Res}(H^2(G, M').$$

Proof. We begin with a). As before, we have the exact sequence $\mathrm{Ext}(\mathbb{Z}[G/H'], M) \to \mathrm{Ext}(J, M) \to \mathrm{Ext}^2(I, M) \to \mathrm{Ext}^2(\mathbb{Z}[G/H'], M)$. In addition, we have the exact sequence $\mathrm{Ext}^2(\mathbb{Z}, M) \to \mathrm{Ext}^2(\mathbb{Z}[G/H], M) \to \mathrm{Ext}^2(I, M) \to \mathrm{Ext}^3(\mathbb{Z}, M) \to \mathrm{Ext}^3(\mathbb{Z}[G/H], M)$. Since $\mathrm{Ext}(\mathbb{Z}[G/H'], M) = H^1(H', M)$, we have $\mathrm{Ext}(J, M) \to \mathrm{Ext}^2(I, M)$ is injective. In a similar way, $\mathrm{Ext}^2(\mathbb{Z}[G/H'], M) \cong H^2(H', M)$, $\mathrm{Ext}^i(\mathbb{Z}[G/H], M) = H^i(H, M)$, and $\mathrm{Ext}^i(\mathbb{Z}, M) \cong H^i(G, M)$. Just as in 0.4, the induced map $H^i(G, M) \to H^i(H, M)$ is restriction. Thus we have exact sequences $H^2(G, M) \to H^2(H, M) \to \mathrm{Ext}^2(I, M) \to H^3(G, M) \to H^3(H, M)$ and $0 \to \mathrm{Ext}(J, M) \to \mathrm{Ext}^2(I, M) \to H^2(H', M)$. The elements of $H^2(H, M)$ which define elements of $\mathrm{Ext}(J, M)$ are precisely the elements in the kernel of $H^2(H, M) \to \mathrm{Ext}^2(I, M) \to H^2(H', M)$. If A' is this kernel, then the injectivity of $\mathrm{Ext}(J, M) \to \mathrm{Ext}^2(I, M)$ shows that a) holds for A'. Thus to prove a) it suffices to observe:

Lemma 0.8. *The composition $h : H^2(H, M) \to \mathrm{Ext}^2(I, M) \to \mathrm{Ext}^2(\mathbb{Z}[G/H'], M) = H^2(H', M)$ is the map taking β to the restriction $(\beta - g(\beta))|_{H'}$.*

Proof. Let $f : \mathbb{Z}[G/H'] \to \mathbb{Z}[G/H]$ be the induced map taking the canonical generator to $u_1 - u_{g(1)}$. Then f induces $f' : \mathrm{Hom}(\mathbb{Z}[G/H], M) \to \mathrm{Hom}(\mathbb{Z}[G/H'], M)$. f', in turn,

induces $f'^* : H^2(G, \mathrm{Hom}(\mathbb{Z}[G/H], M)) \to H^2(G, \mathrm{Hom}(\mathbb{Z}[G/H'], M))$. The map h we are analyzing is the composition of $\phi_{H'} f'^* \phi_H^{-1}$ where $\phi_H : H^2(G, \mathrm{Hom}(\mathbb{Z}[G/H], M)) \to H^2(H, M)$ and $\phi_{H'} : H^2(G, \mathrm{Hom}(\mathbb{Z}[G/H'], M) \cong H^2(H', M)$ are the isomorphisms from Shapiro's Lemma ([B] p. 73 and p. 80). Note that as an H module, $\mathbb{Z}[G/H] = \mathbb{Z} \oplus \mathbb{Z}[H/H \cap gHg^{-1}]$. In particular, $H^2(H, \mathrm{Hom}(\mathbb{Z}[G/H], M)) = H^2(H, M) \oplus H^2(H \cap gHg^{-1}, M))$. Suppose $\phi_H^{-1}(\beta_H) = \beta_G \in H^2(G, \mathrm{Hom}(\mathbb{Z}(G/H), M))$. The restriction $\beta_G|_H$ is easily seen to be

$$(\beta_H, \beta') \in H^2(H, M) \oplus H^2(H \cap gHg^{-1}, M) = H^2(H, \mathrm{Hom}(\mathbb{Z}[G/H], M)).$$

Here β' is the restriction of $g(\beta_H)$ to $H \cap gHg^{-1}$.

The Shapiro Lemma ([B] p. 80) isomorphism $\phi_H : H^2(G, \mathrm{Hom}(\mathbb{Z}[G/H'], M)) \cong H^2(H', M)$ is a composition $H^2(G, \mathrm{Hom}(\mathbb{Z}[G/H'], M)) \to H^2(H', \mathrm{Hom}(\mathbb{Z}[G/H'], M)) \to H^2(H', M)$ where the first map is restriction and the second map, call it $\rho^*_{H'}$, is induced by the canonical H' morphism $\rho_{H'} : \mathrm{Hom}(\mathbb{Z}[G/H'], M) \to M$. Thus in computing $h(\beta_H) = \phi_{H'}(f'^*(\beta_G)$, we can first restrict β_G to H'. That is, $\phi_{H'}(f'^*(\beta_G) = {\rho'_{H'}}^*(f'^*|_H(\beta_G|_{H'}))$ where $f'^*|_{H'}$ is the induced map on H' cohomology groups. Since $H' \subset H \cap gHg^{-1} \subset H$, the restriction, $\beta_G|_{H'}$, can be written $(\beta_H|_{H'}, g(\beta)|_{H'})$. View f, and hence f' as H' module maps (so $f'^*|_{H'}$ can be written simply as f'^*). It remains to compute $(\rho_{H'} \circ f'))^*(\beta_H|_{H'}, g(\beta)|_{H'}$.

Let $\rho : \mathbb{Z} \to \mathbb{Z}[G/H']$ be the H' map sending $1 \in \mathbb{Z}$ to the canonical H' fixed generator of $\mathbb{Z}[G/H']$. Then ρ induces the $\rho_{H'} : \mathrm{Hom}(\mathbb{Z}[G/H'], M) \to \mathrm{Hom}(\mathbb{Z}, M) = M$ mentioned above. $\rho_{H'} \circ f' : \mathrm{Hom}(\mathbb{Z}[G/H], M) \to \mathrm{Hom}(\mathbb{Z}, M) = M$ is induced by $f \circ \rho : \mathbb{Z} \to \mathbb{Z}[G/H] = \mathbb{Z} \oplus \mathbb{Z}[H/(H \cap gHg^{-1})]$ which we compute sends $1 \in \mathbb{Z}$ to $(1, -u_{gH}) \in \mathbb{Z} \oplus \mathbb{Z}[H/(H \cap gHg^{-1})]$. Now u_{gH} is the canonical generator of $\mathbb{Z}[H/(H \cap gHg^{-1}]$, and this proves that $(\rho_{H'} \circ f'))^*(\beta_H|_{H'}, g(\beta)|_{H'} = \beta_H|_{H'} - g(\beta)|_{H'}$. Thus a) is proven.

We next consider b). By assumption, $H^3(G, M) \to H^3(H, M)$ has 0 kernel and so $H^2(H, M) \to \mathrm{Ext}^2(I, M)$ is surjective. If $\alpha \in \mathrm{Ext}(J, M)$, then α maps to, say, $\alpha' \in \mathrm{Ext}^2(I, M)$ which is the image of some $\beta \in H^2(H, M)$. Since α' maps to 0 in $H^2(H', M)$, $\beta \in A$ and b) is proven.

Finally we turn to showing part c) of 0.7. Let $\alpha \in \mathrm{Ext}(J, M)$ correspond to $0 \to M \to M' \to J \to 0$ and let $\beta \in H^2(H, M)$ be the preimage of α. Of course, the canonical map $\mathrm{Ext}(J, M) \to \mathrm{Ext}(J, M')$ maps α to 0. It follows that the image of β in $H^2(H, M')$ is also in the image of some $\delta \in H^2(G, M')$.

We have the diagram:

$$\begin{array}{ccccccc}
H^2(G,M) & \longrightarrow & H^2(G,M') & \longrightarrow & H^2(G,J) & \longrightarrow & H^3(G,M) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^2(H,M) & \longrightarrow & H^2(H,M') & \longrightarrow & H^2(H,J) & \longrightarrow & H^3(H,M)
\end{array}$$

Suppose $\beta' \in H^2(H, M)$ is such that the image of β' in $H^2(H, M')$ is also in the image of some $\delta' \in H^2(G, M')$. We need to show that β' is a power of β modulo $\mathrm{Res}(H^2(G, M)) + \delta(H^1(H, J))$.

Let $\epsilon, \epsilon' \in H^2(G, J)$ be the image of δ, δ' respectively. We claim it suffices to show that ϵ' is a power of ϵ. To prove the claim, assume $\epsilon' = \epsilon^r$. Then $\delta' = \delta^r\chi$ where χ is the image of $\mu \in H^2(G, M)$. Modifying β', δ' by the image of μ^{-1}, we may assume $\delta' = \delta^r$. The result is now clear.

Thus we are reduced to showing ϵ' is a power of ϵ. Tracing through the above diagram, ϵ' maps to 0 in $H^2(H, J)$. Consider the exact sequence $H^1(G, I) \to H^2(G, J) \to H^2(G, \mathbb{Z}[G/H'])$. The map $H^2(H', \mathbb{Z}) \cong H^2(G, \mathbb{Z}[G/H']) \to H^2(H, \mathbb{Z}[G/H'])$ is injective because the inverse of the isomorphism is restriction to H' followed by projection to $\mathbb{Z}$, and one can factor the restriction to H' step through restriction to H. It follows that ϵ' maps to 0 in $H^2(G, \mathbb{Z}[G/H'])$ and hence is in the image of $H^1(G, I)$. But tracing through the equivalences, it is easy but tedious to see that ϵ is the image of a generator of $H^1(G, I)$, proving c). ∎

Let us take the construction in 0.7 and apply it to fields. In 0.7, let M have the form L^* where L/F is a G Galois extension. Recall that an extension $\gamma : 0 \to L^* \to M' \to J \to 0$ defines a "twisted" action of G on $L[J]$ and the field of fractions $L(J)$. If this extension is defined by $\beta \in H^2(H, L^*) = \mathrm{Br}(L/L^H)$, we write $L[J]$ and $L(J)$ with this twisted action as $L_\beta[J]$ and $L_\beta(J)$ respectively. Note that we can form $L_\beta(J)$ for any β with $(\beta - g(\beta))|_{H'} = 0$. Also note that if β is the image of $\alpha \in \mathrm{Br}(L^G)$ then since L splits β, L must split α. That is, α defines an element in $H^2(G, L^*)$. Translating 0.7. we have:

Proposition 0.10. *The extension of β to $\mathrm{Br}(L_\beta(J)^H)$ is in the image of $\mathrm{Br}(L_\beta(J)^G)$. If β is in the image of $\mathrm{Br}(L^G)$, then $L_\beta[J] \cong L[J]$ the isomorphism preserving G actions and $L[J]$ having untwisted action.*

Section One: Reducing the finite group

We begin by recalling the definition of the projective orthogonal and symplectic groups. Of course, $PGL_n(F) = GL_n(F)/F^*$ is the quotient of $GL_n(F)$ modulo its center. The orthogonal group $O_n = O_n(F)$ is the subgroup of $GL_n(F)$ where $AA^T = I$ and T refers to the transpose. We define $PO_n = PO_n(F)$ to be the image of $O_n(F)$ in $PGL_n(F)$. Note that this means $PO_n(F)$ is not necessarily the F points of the corresponding algebraic group scheme. To avoid this technicality we would have to introduce GO_n, the group of so called similitudes (e.g. [K-T] p. 153). However, the subgroup $PO_n(F)$, as we have defined it, is Zariski dense in the $\bar{F}$ points (i.e. over the algebraic closure) and in considering invariant rings or fields this issue is therefore irrelevant.

Next we recall the definition of the symplectic group. The symplectic involution J_1 on 2×2 matrices is defined as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} .$$

We identify $M_{2m}(F)$ with $M_2(F) \otimes_F M_m(F)$ so that the matrix idempotent $e_{11} \otimes e_{ii}$ is $e_{2i-1,2i-1}$ and $e_{22} \otimes e_{ii}$ is $e_{2i,2i}$. The symplectic involution J_m on M_{2m} can then be

described as $J_1 \otimes T$ where T is the transpose on $M_m(F)$. From now on, we write J_m as J. Of course, Sp_n is the group of matrices $A \in M_{2m}(F)$ such that $A^J A = AA^J = I_{2m}$. We define $PSp_n(F)$ to be the image of Sp_n in $PGL_n(F)$, so $PSp_n(F) = Sp_n(F)/\{I, -I\}$. Once again $PSp_n(F)$ are not the F points of the group scheme, and once again it does not matter.

Let $V = M_n(F) \oplus M_n(F)$ with the natural diagonal action of $PGL_n(F)$. Then $F(V)^{PGL_n}$ is the center $Z = Z(F, n, 2)$ of the generic division algebra of degree n in 2 variables which we write as $UD = UD(F, n, 2)$. As remarked before, combining [P2] p. 377-78 and [R] p. 184 we have that $F(V)^{PO_n}$ and $F(V)^{PSp_n}$ are the centers of the generic algebras with appropriate involution. Combining this with [BS] p. 112, , we have the following.

Theorem 1.1. *a) If n is even, $F(V)^{PSp_n}/F(V)^{PGL_n} = F(V)^{PSp_n}/Z$ is the function field of the Severi Brauer variety of an algebra A/Z where A has degree $n(n-1)/2$ and is Brauer equivalent to $UD \otimes_Z UD$.*

b) $F(V)^{PO_n}/F(V)^{PGL_n} = F(V)^{PO_n}/Z$ is the function field of the Severi Brauer variety of an algebra A/Z where A has degree $n(n+1)/2$ and is Brauer equivalent to $UD \otimes_Z UD$.

Recall that if B/K is any central simple algebra, we write $K(B)$ to mean the function field of the Severi-Brauer variety of B. That is, $K(B)$ is the Amitsur generic splitting field of B. By e.g. [LN] 13.12, $K(M_r(B))$ is rational over $K(B)$. In particular, let D/Z be the division algebra Brauer equivalent to $UD \otimes_Z UD$, which it is not hard to see has degree $m = n/2$ or n depending on whether n is even or odd. One result we desire is now easy.

Theorem 1.2. *Suppose n is odd. Then $F(V)^{PO_n}$ is rational over F.*

Proof. By the result 0.3 b) of Tregub, $Z(D) \cong Z(UD)$. By 0.6, $Z(UD)/F$ is rational. ∎

We can now assume $n = 2m$ is even. Using 1.1, both $F(V)^{PO_n}$ and $F(V)^{PSp_n}$ are rational over $Z(D)$ of degree $n(n+1)/2 - m$ and $n(n-1)/2 - m$ respectively. In particular, to prove rationality or stable rationality results for $F(V)^{PSp_n}$ and $F(V)^{PO_n}$, it suffices to consider the case $PSp_n(F)$.

Procesi showed that $Z(F, n, 2) = F(V)^{PGL_n}$ can be written as a multiplicative invariant field $F(M)^{S_n}$ where M is a lattice whose definition we will recall later. Here S_n is the symmetric group in n letters, and so is the Weyl group of PGL_n ([P] or e.g. [LN] p. 109). In this section we will recall the parallel argument for PSp_n, and so get $F(V)^{PSp_n}$ as a multiplicative invariant field of the Weyl group, W, of PSp_n. In addition, we will prove an analogue of 1.1 where Z is replaced by $F(M)^W$. This is a significant improvement, as W is smaller than S_n and has an abelian normal subgroup we will use.

To this end, let $T_{PGL} = T_{GL}/F^*$ where $T_{GL} \subset GL_n(F)$ is the group of diagonal matrices. Of course T_{PGL} is a maximal torus of $PGL_n(F)$. Let $S_n \subset PGL_n$ be the group of permutation matrices, so $N_{PGL_n} = T_{PGL_n} S_n$ is the normalizer of T_{PGL_n} and the inclusion $S_n \subset N_{PGL_n}$ induces $S_n \cong N_{PGL_n}/T_{PGL_n}$.

Let $T_{Sp} \subset Sp_n$ be the subgroup of Sp_n of diagonal matrices. The condition defining Sp_n implies that any such diagonal matrix has the form $(a, a^{-1}, b, b^{-1}, \ldots, c, c^{-1})$ down the diagonal. Set $T_{PSp} \subset PSp_n$ to be the image. T_{PSp} is a maximal torus of PSp_n.

Let N_{PSp} be the normalizer of T_{PSp} in PSp_n. Then $W = N_{PSp}/T_{PSp}$ is the Weyl group and can be described as follows. Writing $M_{2m}(F) = M_2(F) \otimes M_m(F)$, consider $\tau_i = \tau \otimes e_{ii} + \sum_{j \neq i} I_2 \otimes e_{jj}$ where:

$$\tau = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Note that $\tau_i^2 \in T_{Sp}$, and that τ_i normalizes T_{Sp}. Let $A \subset W$ be generated by the τ_i, so that $A \cong \mathbb{Z}/2\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/2\mathbb{Z}$ (m times). Let $S_m \subset M_m(F)$ be the group of permutation matrices embedded in $M_{2m}(F)$ via $\sigma \to 1 \otimes \sigma$, and identifying S_m with this image, S_m normalizes T_{Sp}. One can show that $W \cong A \rtimes S_m$ where S_m has the obvious permutation action on the τ_i. Said differently, $W \subset S_{2m}$ is the subgroup preserving the partition $\mathcal{P} = \{\{1, 2\}, \{3, 4\}, \ldots, \{2m - 1, 2m\}\}$.

Having the definition of W in front of us, let us define some objects involving W. Let $p'_i : A \to \mathbb{Z}/2\mathbb{Z}$ be defined by $p'_i(\tau_j) = 0$ for $j \neq i$ and $p'_i(\tau_i) = 1 + 2\mathbb{Z}$. Let $S_{m-1} \subset S_m$ be the subgroup fixing both elements in $\{1, 2\}$, and set $H = A \rtimes S_{m-1}$. Note that H is the stabilizer in W of the set $\{1, 2\}$. By setting $p_1(S_{m-1}) = 0 + 2\mathbb{Z}$, p'_1 extends to $p_1 : H \to \mathbb{Z}/2\mathbb{Z}$.

The embedding $PSp_n \subset PGL_{2m}(F)$ induces the commutative diagram:

$$\begin{array}{ccc} T_{Sp} & \subset & T_{GL} \\ \downarrow & & \downarrow \\ T_{PSp} & \subset & T_{PGL} \end{array} \tag{1}$$

We also have $N_{PSp} \subset N_{PGL}$. Since $T_{PGL} \cap N_{PSp} = T_{PSp}$, we have an induced embedding $W \subset S_{2m}$ which is precisely the one described above. Finally, note that we can define an intermediate group $N_{PSp} \subset N' \subset N_{PGL}$ such that $N' \supset T_{PGL}$ and $N'/T_{PGL} = W$.

As a module over PSp_n, $M_n(F)$ is a direct sum of the spaces of symmetric and skew symmetric matrices. Let M^- be the later. Of course, M^- is the Lie algebra of PSO_n. Let $\Delta \subset M^-$ be the sub vector space of diagonal matrices, necessarily of the form $(a, -a, b, -b, \ldots, c, -c)$ down the diagonal. Note that Δ is preserved by N_{PSp}.

Since M^- is the Lie algebra of G, there is a invariant Zariski open subset $U \subset M^-$ such that, for all $x \in U$, $Gx \cap \Delta \neq \emptyset$ and if $gx, g'x \in D$, then there is an $h \in N_{PSp}$ such that $hgx = g'x$.

Let V^- be the PSp_n representation $M^- \oplus M_n(F)$. Then by A.1, we can write $F(V^-)^{PSP_n} = F(\Delta \oplus M_n(F))^{N_{PSp}}$. Note that T_{PGL} acts trivially on Δ, so $V^- \subset V$ is preserved by N' and the action of N_{PSp} on V^- extends to N'.

Set $K = F(\Delta)$ and view $F(\Delta \oplus M_n(F))$ as $K(M_n(F))$. If the $y_{ij} \in K(M_n(F))$ correspond to the standard basis of $M_n(F)$, then $K(M_n(F))$ is the field of fractions of $K[Y']$ where $Y' \subset K(M_n(F))$ is the multiplicative subgroup generated by the y_{ij}. The

elements of Y' (i.e. monomials in the y_{ij}) in $K[Y']$ form a basis of T_{PGL} and hence T_{PSp} eigenvectors. We next evaluate $K[Y']^{T_{PGL}}$ and $K[Y']^{T_{PSp}}$.

The character group $\mathrm{Hom}_F(T_{PGL}, F^*)$ is a sublattice of $\mathrm{Hom}(T_{GL}, F^*) = \mathbb{Z}[S_{2m}/S_{2m-1}]$. More precisely, let $d_i : T_{GL} \to F^*$ be the projection on the i^{th} diagonal entry. Then the d_i form a basis of $\mathrm{Hom}(T_{GL}, F^*)$ permuted by S_{2m} in the obvious way. $\mathrm{Hom}(T_{PGL}, F^*)$ is the sublattice generated by all elements of the form $d_i - d_j$. Note that $d_i - d_j$ is precisely the character associated to the eigenvector y_{ij}. Set $I_{2m} = \mathrm{Hom}(T_{PGL}, F^*)$. Then we have a surjective S_{2m} morphism $Y' \to I$ associating each monomial to its character, and we set Y to be the kernel. It is clear that $K[Y']^{T_{PGL}}$ is spanned by the T_{PGL} fixed monomials and so $K[Y']^{T_{PGL}} = K[Y]$.

To compute $K[Y']^{T_{PSp}}$ we begin by computing $M_{Sp} = \mathrm{Hom}(T_{Sp}, F^*)$ as an image of $\mathbb{Z}[S_{2m}/S_{2m-1}]$ under restriction. From the structure of T_{Sp} we have that the character $f_i = d_{2i} + d_{2i-1}$ is trivial on T_{Sp}. It is clear that $M_{Sp} = \mathbb{Z}[S_{2m}/S_{2m-1}]/\mathbb{Z}[S_m/S_{m-1}]$ where $\mathbb{Z}[S_m/S_{m-1}]$ is the sublattice generated by the f_i. If we set $M_{PSp} \subset M_{Sp}$ to be $\mathrm{Hom}(T_{PSp}, F^*)$, it is clear from (1) that $M_{PSp} = I_{2m}/(\mathbb{Z}[S_m/S_{m-1}] \cap I_{2m})$. Furthermore, one can compute that $\mathbb{Z}[S_m/S_{m-1}] \cap I_{2m}$ is the lattice generated by all $f_i - f_j$ which we write as I_m. Note that W acts naturally on all these lattices via its action on all the tori. Set $H' = W \cap S_{2m-1}$ so as a W lattice, $\mathbb{Z}[S_{2m}/S_{2m-1}] = \mathbb{Z}[W/H']$. A acts trivially on $\mathbb{Z}[S_m/S_{m-1}]$ and the induced action by $S_m = W/A$ is just the permutation action indicated. That is, $\mathbb{Z}[S_m/S_{m-1}]$ is the lattice $\mathbb{Z}[W/H]$ where $H \subset W$ is generated by S_{m-1} and A. All together we have the following diagram of W lattices corresponding to the diagram of tori (1):

$$\begin{array}{ccc} \mathbb{Z}[W/H']/\mathbb{Z}[W/H] & \longleftarrow & \mathbb{Z}[W/H'] \\ \uparrow & & \uparrow \\ I_{2m}/I_m & \longleftarrow & I_{2m} \end{array}$$

Once again, there is a W morphism $Y' \to I_{2m}/I_m$ taking each monomial to its T_{PSp} character. Clearly this morphism is just the composition $Y' \to I_{2m} \to I_{2m}/I_m$. Furthermore, if Y_2 is the kernel of $Y' \to I_{2m}/I_m$, then $K[Y']^{T_{PSp}} = K[Y_2]$. We have begun the proof of:

Proposition 1.3. *a)* $K[Y']^{T_{PGL}} = K[Y]$; $K[Y']^{T_{PSp}} = K[Y_2]$

b) $Y \subset Y_2$ *and* $Y_2/Y \cong I_m$

c) $K(Y')^{T_{PGL}} = K(Y)$; $K(Y')^{T_{PSp}} = K(Y_2)$

Proof. We have already shown a), and b) follows from the fact that Y_2 is the inverse image of I_m under the map $Y' \to I_{2m}$. As for c), this follows from the standard:

Lemma 1.4. *Suppose $\mathcal{G} \subset GL_F(V')$ is an algebraic group with $\mathcal{G}_0 \subset \mathcal{G}$ the connected component of the identity. Let $U' \subset V'$ be a $\mathcal{G}$ invariant basic Zariski open subset (including the case $U' = V'$). If $\tau : \mathcal{G}_0 \to F^*$ is any character, assume there is a $u \in F[U']$ such that $\eta(u) = \tau(\eta)^{-1}u$ for all $\eta \in \mathcal{G}_0$. Then the field of fractions $q(F[U']^{\mathcal{G}})$ is $F(V')^{\mathcal{G}}$.*

Proof. Write $F[U'] = F[V'](1/d)$. Since $F[V']$ is a UFD, $F[U']^*/F^*$ is a free abelian group with basis corresponding to the primes dividing d. For all $\eta \in \mathcal{G}_0$, $\eta(d) \in dF^*$ so $\mathcal{G}_0$ must permute the primes dividing d. That is, $\mathcal{G}_0$ acts trivially on $F[U']^*/F^*$.

Since $F[U']$ is a UFD, if $\alpha \in F(V')$ is $\mathcal{G}$ invariant, we may assume $\alpha = f/g$ where f, g have no common factors. Since $\mathcal{G}/\mathcal{G}_0$ is finite, it suffices to find f, g which are $\mathcal{G}_0$ invariant. If $\eta \in \mathcal{G}_0$, $g\eta(f) = \eta(g)f$. Since f, g have no common factors, $\eta(f) = f\tau(\eta)$ and $\eta(g) = g\tau(g)$ where $\tau(g) \in F[U']^*$. Since $\mathcal{G}_0$ acts trivially on $F[U']^*/F^*$, the map $\mathcal{G}_0 \to F[U']^*/F^*$ induced by τ must be a homomorphism. Since $\mathcal{G}_0$ is connected, it follows that $\tau(g) \in F^*$ and $\tau : \mathcal{G}_0 \to F^*$ is a character. Choose $u \in F[U']$ as in the given. Then $\alpha = uf/ug$ and $uf, ug \in F[U']^{\mathcal{G}_0}$. ■

Having described the T_{PSp} invariant field, the first part of the next proposition is clear. As for the rest, note that it says $F(V)^{PSp_n}$ is "too big" and the important information resides in a smaller field. Not only do we make it smaller by substituting $F(V^-)$ for $F(V)$, but we observe the "Δ" part is also irrelevant.

Proposition 1.5. *$F(V^-)^{PSp_n} = K(Y_2)^W$. $F(V)^{PSp_n}/F(V^-)^{PSp_n}$ is rational of transcendence degree $n(n-1)/2$. $K(Y_2)^W/F(Y_2)^W$ is rational of degree m, so all together $F(V)^{PSp_n}/F(Y_2)^W$ is rational of degree $2m^2$.*

Proof. $F(V)^{PSp_n} = F(M^+ \oplus V^-)^{PSp_n}$ where M^+ is the submodule of M_n consisting of J symmetric matrices. The second statement now follows from A.3. $K = F(\Delta)$ so $K(Y_2) = F(Y_2)(\Delta)$. Since W acts linearly on Δ, the rest follows from the result of Endo-Miyata (0.2). ■

Of course, we have exact sequences $0 \to I_m \to \mathbb{Z}[W/H] \to \mathbb{Z} \to 0$ and $\beta_2 : 0 \to Y \to Y_2 \to I_m \to 0$. By 0.4, this second sequence is associated to an element of $\gamma_2 \in H^2(W, Y)$ split by H. More precisely, it is associated with the element $\delta_2(\alpha_m)$ where $\alpha_m \in H^1(W, I_m)$ is the canonical generator and δ_2 is the boundary of the long exact sequence associated to β_2. Of course Y is also part of the S_{2m} sequence $\beta : 0 \to Y \to Y' \to I_{2m} \to 0$ and this defines $\gamma \in H^2(S_{2m}, Y)$ as $\delta(\alpha_{2m})$ where $\alpha_{2m} \in H^1(S_{2m}, I_{2m})$ is the canonical generator and δ is the boundary associated with β. There is a W embedding $I_m \to I_{2m}$ defined above and direct computation shows that the image of α_m is twice the restriction of α_{2m} to W. Thus γ_2 is twice the restriction of γ to W.

The standard argument, which parallels the one proving 1.5, shows that $Z(F, n, 2) = F(V)^{PGL_n} = F(X \oplus Y)^{S_n}$ where $X = \mathbb{Z}[S_n/S_{n-1}]$. Another direct computation shows that γ is the cocycle associated to the generic division algebra $UD(F, n, 2)$. Since Δ has rank m and X has rank $n = 2m$, $F(X \oplus Y)^W/F(Y)^W$ is rational of transcendence degree $2m$. It follows that $F(Y)^W$ can be thought of as the center of a generic division algebra with maximal subfield having W as Galois group. This division algebra is described by the restriction of γ to W and we call it D_γ. Let D_2 be the division algebra of degree m Brauer equivalent to $D_\gamma \otimes_{F(Y)^W} D_\gamma$.

The sequence $0 \to Y \to Y_2 \to I_m \to 0$ induces the sequence $\beta' : 0 \to F(Y)^* \to F(Y)^*Y_2 \to I_m \to 0$ and the field $F(Y_2)$ with its W action can be thought of as $F(Y)_{\beta'}(I_m)$. Thus by 0.5 we have the following.

Lemma 1.6. *$F(Y_2)^W = F(Y)^W(D_2)$. That is, $F(Y_2)^W/F(Y)^W$ is the function field of the Severi Brauer variety of D_2.*

Set U to be the F representation $F[W/H']$ of W, where we recall that $H' = W \cap S_{2m-1}$. With its W action, $F(X)$ is just $F(U)$. In particular, by Endo-Miyata (0.2)) again, $F(X \oplus Y)^W/F(Y)^W$ is purely transcendental of degree $2m$. Finally, as we mentioned before, for any central simple algebra A/L, $L(M_r(A))$ is purely transcendental over $L(A)$. Using these facts, and adding the appropriate number of indeterminants to the fields involved, we have the following.

Theorem 1.7. *Let A be the central simple algebra of degree $n(n-1)/2$ in the Brauer class of $\Delta(F(Y)/F(Y)^W, W, \gamma^2)$. Let A' be the central simple algebra in the same Brauer class of degree m.*

a) $F(V)^{PSp_n} \cong F(X \oplus Y)^W(A)$.

b) $F(V)^{PSp_n}$ is rational over $F(Y)^W(A')$ of degree $2m^2$.

Note that a) above is almost the same as 1.1 a) except the group $S_n = S_{2m}$ has been replaced by W. This is what is meant by "reducing the group" in the title of this section.

Section Two: More about lattices

We want to look further at the W lattice Y defined in section one. To begin with, we note that as a lattice over W, Y' is the direct sum of two sublattices, we call Y'_D and Y'_O. To be exact, Y'_D is spanned by the y_{ij} where i, j are in the same element of the partition $\mathcal{P} = \{\{1,2\}, \ldots, \{2m-1, 2m\}\}$, and Y'_O is the span of the rest of the y_{ij}. Graphically, the y_{ij} in Y'_D correspond to matrix entries on the 2×2 matrix diagonal of the matrix (y_{ij}) and the $y_{ij} \in Y'_O$ correspond to the off diagonal entries. Also note that $Y'_O = \mathbb{Z}[W/H'']$ where H'' is the stabilizer of y_{13}. That is, H'' is $A' \rtimes S_{m-2}$ where $S_{m-2} \subset S_m$ is the stabilizer of $\{1,2,3,4\}$ and $A' \subset A$ is the kernel of the projection $p'_1 \oplus p'_2 : A \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Consider the image of Y'_D under the morphism $Y' \to I_{2m-1}$. Clearly this image is generated by $d_i - d_j$ where i, j are in the same element of the partition $\mathcal{P}$. We wish to describe this image further. To this end, let $H = A \rtimes S_{m-1}$ be the subgroup fixing $d_1 + d_2$ just as in section one. Then the image in question is clearly isomorphic to $\mathbb{Z}^-[S_m/S_{m-1}] = \mathrm{Ind}_H^W(\mathbb{Z}^-)$ where $\mathbb{Z}^- = \mathbb{Z}(d_1 - d_2)$. Of course $\mathbb{Z}^-$ is the rank one lattice associated to the homomorphism $p_1 : H \to \{1, -1\} \cong \mathbb{Z}/2\mathbb{Z}$. Set $H' \subset H$ to be the kernel of this map, so $H' = W \cap S_{2m-1}$ as in section one.

It is obvious that the cokernel of $\mathbb{Z}^-[S_m/S_{m-1}] \to I_{2m-1}$ is I_m where I_m fits into

the sequence $0 \to I_m \to \mathbb{Z}[W/H] \to \mathbb{Z}$. All together we have the diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & Y_O & \to & Y'_O & \to & I_m & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & Y & \to & Y' & \to & I_{2m-1} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & Y_D & \to & Y'_D & \to & \mathbb{Z}^-[S_m/S_{m-1}] & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Let us consider further some of the lattices appearing above. Define $M \subset Y_D$ to be the H sublattice generated by $y_{11}, y_{22}, y_{12} + y_{21}$. Of course, as an H module, $M = \mathbb{Z}[H/H'] \oplus \mathbb{Z}$. It is then easy to see that $Y_D = \mathrm{Ind}_H^W(M)$. In particular Y_D is a permutation lattice. Thus $H^1(W', Y_D) = 0$ for any subgroup of $W' \subset W$. Shapiro's Lemma ([B] p. 73, 80) says $H^3(W, Y_D) \cong H^3(H, M)$ where the map is restriction to H followed by projection onto M. If follows that $H^3(W, Y_D) \to H^3(H, Y_D)$ is injective.

Note that $W = H \cup HgH$ where $g(1) = 3$ and $g(2) = 4$. Then $H \cap gHg^{-1}$ contains the group H'' defined above. We have $H^1(H'', Y_D) = 0$. Since $Y'_O = \mathbb{Z}[W/H'']$ and the map $Y'_O \to I_m$ takes the H'' fixed generator to $g(1) - 1$, Y_O is a lattice of the form of J in 0.7. In all, 0.7 applies to the extension $0 \to Y_D \to Y \to Y_O \to 0$. We label this extension as β', so if we set $L' = F(Y_D)$, then there is a W action preserving field isomorphism $F(Y) \cong L'_{\beta'}(Y_O)$.

Of course the above observation is most useful if we compute the element $\beta'' \in H^2(H, Y_D)$ associated to β'. Using 0.7 c) we can begin to do this by computing the kernel of $H^2(H, Y_D)/(\mathrm{Res}(H^2(W, Y_D)) + \delta(H^1(H, Y_O))) \to H^2(H, Y)/\mathrm{Res}(H^2(W, Y))$. Recall that $\mathbb{Z}^-[S_m/S_{m-1}]$ is spanned by all $d_i - d_j$ where i, j are in the same element of $\mathcal{P}$. Let $\alpha \in H^1(H, \mathbb{Z}^-[S_m/S_{m-1}])$ be given by the cocycle $h \to h(d_1) - d_1$. Then clearly the image of α in $H^1(H, I_{2m-1})$ is the image of of the canonical generator of $H^2(W, I_{2m-1})$. Let $\beta \in H^2(H, Y_D)$ be the image of α under the boundary of the long exact sequence. Then the naturality of this boundary shows that the image of β in $H^2(H, Y)$ is the restriction of the canonical element $\gamma \in H^2(W, Y)$.

It will be convenient to describe this β precisely. To begin with, $Y_D = \mathrm{Ind}_H^W(M)$ so $H^2(H, Y_D) = H^2(H, M) \oplus H^2(H \cap gHg^{-1}, g(M))$ (e.g. [B] p. 69). One can compute that $\beta = (\beta_H, 0)$ where $\beta_H \in H^2(H, M)$ is the inflation, via p_1, of the nontrivial element of $H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}(y_{12} + y_{21}))$.

We will need to define a similar element in $H^2(H \cap gHg^{-1}, gM)$. Note that $g(M)$ can be viewed as the submodule spanned by y_{33}, y_{44}, and $y_{34} + y_{43}$. We define $p_2 : gHg^{-1} \to$

$\mathbb{Z}/2\mathbb{Z}$ to be the g translation of p_1, and we note p_2 restricts to p_2' on A. We also use p_2 to denote the restriction to $H \cap gHg^{-1}$. Then we set $\beta_{gH} \in H^2(H \cap gHg^{-1}, g(M)) \subset H^2(H, Y_D)$ to be the inflation via p_2 of the nontrivial element of $H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}(y_{34}+y_{43}))$.

We will show $\beta'' = \beta$ in some cases. To do this using 0.7 we compute $\delta(H^1(H, Y_O)) \subset H^2(H, Y_D)$ which first requires computing $H^1(H, Y_O)$. But this last cohomology group is the image of I_m^H under the boundary for the top horizontal sequence above. In fact, if $f_i \in \mathbb{Z}[S_m/S_{m-1}]$ is the image of d_{2i-1} and d_{2i} in $\mathbb{Z}[S_{2m}/S_{2m-1}]$, then $\eta' = (m-1)f_1 - \sum_{i \neq 1} f_i$ generates I_m^H. Let η be the image of η' in $H^1(H, Y_O)$. Since we ultimately only need the case m is odd we assume this and compute that η has image $(0, \beta_{gH}) \in H^2(H, Y_D)$.

We avoid further tedious cohomology details and just assert that β is not in the image of $H^2(W, Y_D)$ but (β_H, β_{gH}) is. In particular $0 \to Y_D \to Y \to Y_O \to 0$ is not split. By 0.7 c), β and β'' differ by an element of $H^2(W, Y_D) + \delta(H^1(H, Y_O))$. Since $\beta + \mu$ is in $\mathrm{Res}(H^2(W, Y_D))$, it follows that $\beta - \beta'' \in \mathrm{Res}(H^2(W, Y_D))$ and we might as well take $\beta'' = \beta$.

Theorem 2.1. *The sequence $0 \to Y_D \to Y \to Y_O \to 0$ is associated, as in 0.7, with the element $\beta \in H^2(H, Y_D)$ defined above.*

In algebra terms, β corresponds to the quaternion algebra $B = \Delta(F(Y_D)^{H'}/F(Y_D)^H, y_{12}y_{21})$ over $F(Y_D)^H$. To see this recall that $H' = W \cap S_{2m-1}$ can also be described as the kernel of p_1. The gist of 2.1 is that the extension $F(Y)^W/F(Y_D)^W$ is a generic extension forcing β to be in the image of $\mathrm{Br}(F(Y)^W)$.

Section three: A cheaper way

In the last section, we described a generic way of forcing a Brauer group element to be in the image of restriction. In this section we describe a "cheaper" way, by which we mean a way that requires smaller transcendence degree. We will study its properties and study its connection with a construction involving generic matrices. But we approach this whole subject from the generic division algebra side, and in fact will start by considering a natural invariant theory problem involving generic matrices.

To begin, we recall with slightly different emphasis an argument from [S1] (or [LN] p. 113). Let D/F be a central simple algebra of degree n, and $UD = UD(F, r, s)$ the generic division algebra of degree r in s variables. We will also write UD as $UD(F, P_1, \ldots, P_s)$ where the P_i are the generic matrices generating UD. Let $Z = Z(F, r, s)$ be the center of UD which we also write as $F(P_1, \ldots, P_s)$. Note that we avoid using X_i, Y_i because in a future argument these P_i will be 2×2 generic matrices and not $n \times n$.

Recall that for A/K any central simple algebra, $K(A)$ is the generic splitting field of A, or equivalently, $K(A)$ is the field of fractions of the Severi-Brauer variety defined by A. With this notation, set $K = Z(UD \otimes_K (D)^\circ)$ where we ask the reader to recall our tensor product convention from the introduction. Let the u_i be a basis of D/F. In [S1] or [LN] p. 113 we showed that:

Proposition 3.1. *There is a canonical isomorphism $\phi : UD \otimes_Z K \to D \otimes_F K$. K/F is rational with transcendence base a_{ik} where $\phi(P_k) = \sum a_{ik} u_i$.*

By 3.1 we can think of $UD \otimes_Z K$ as $D(P_1, \ldots, P_s)$ where the $P_k = \sum a_{ik} u_i$ and $D(P_1, \ldots, P_s)$ has center $F(a_{ik})$. In particular this applies to the case where D itself is a generic division algebra $UD(F', P, Q)$. We can use an old observation of Procesi to note:

Lemma 3.2. *In the situation of 3.1, suppose $D = UD(F', P, Q)))$ so $F = Z(F', r, 2)$. Then K is the center of $UD(F', P, Q, P_1, \ldots, P_s) \cong UD(F, P, Q) \otimes_Z K$.*

Proof. As above, let the u_i be a basis of $UD(F', P, Q)$ over F. Procesi observed that $F'(P, Q, P_1, \ldots, P_s)$ is rational over $F'(P, Q)$ with transcendence basis the y_{ik} where $P_k = \sum y_{ik} u_i$. That is, 3.2 is a direct consequence of 3.1. ∎

Next we consider an invariant theory question. Let S_m be the symmetric group, and let $UD(F, P_1, \ldots, P_m)$ be the generic division of degree r in m variables with center $F(P_1, \ldots, P_m)$. Abbreviate these algebras or fields as $UD(F, \vec{P})$ and $F(\vec{P})$ respectively. Then S_m acts in the natural way on $UD(F, \vec{P})$ and hence on $F(\vec{P})$ by permuting the P_i's. To be precise, we view S_m as the bijections of $\{1, \ldots, m\}$ and $\sigma(P_i) = P_{\sigma(i)}$ for all $\sigma \in S_m$. Of course, the obvious question is the rationality of the invariant field $F(\vec{P})^{S_m}$. It is not clear how to settle this, but it can be shown that this field is stably rational over the center of the appropriate generic division algebra. The next lemma is all we need in this direction, but see the remark for a bit more.

Lemma 3.3. *Form the rational extension field $F(P, Q, \vec{P}) = F(P, Q, P_1, \ldots, P_m)$ where S_m acts trivially on P, Q. Then $F(P, Q, \vec{P})^{S_m}$ is rational over $F(P, Q)$. Similarly, the*

invariant field of $F(P,Q,P_1,\ldots,P_m,Q_1,\ldots,Q_m) = F(P,Q,\vec{P},\vec{Q})$ with the obvious S_m action is rational over $F(P,Q)$.

Remark. $UD(F,P_1,\ldots,P_m)^{S_m}$ is a division algebra of degree r and so arguments like those of 3.3 show that that $F(P_1,\ldots,P_m,P,Q)^{S_m}$ is rational over $F(P_1,\ldots,P_m)^{S_m}$. It follows that $F(P_1,\ldots,P_m)^{S_m}$ is stably rational.

Proof. Write $F(P,Q,\vec{P}) = F(P,Q,y_{ik})$ where $P_k = \sum y_{ik}u_i$ and u_i is a basis of $UD(F,P,Q)/F(P,Q)$. Clearly the action of S_m on the y_{ik} is just $\sigma(y_{ik}) = y_{i\sigma(k)}$. $F(P,Q,\vec{P})^{S_m}$ is rational over $F(P,Q)$ via the usual fact about invariant fields of S_m with permutation actions. The second sentence follows in the same way. ∎

Set $L = F(P,Q,\vec{P},\vec{Q})$ with the given S_m action, and now set $F_m = L^{S_m}$. We will use 3.2 to analyze the fields L/F_m further. Set $L_k = F(P,Q,P_k,Q_k)$. It is clear that L is the field join $L_1L_2\ldots L_m$ over $F(P,Q)$ with the obvious induced S_m action. Set $UD = UD(F,P,Q)$ so UD has center $F(P,Q)$. Set $UD_k = UD(F(P,Q),P_k,Q_k)$ and let Z_k be the center of UD_k. By 3.2, $L_k = Z_k(UD_k \otimes_{Z_k} (UD)^\circ))$. Let Z be the field join $Z_1\ldots Z_m$ with the obvious S_m action. View $Z = Z_1\ldots Z_m \subset L_1\ldots L_m = L$ in the indicated way so that the S_m actions are compatible.

Denote by $S_{m-1} \subset S_m$ the stabilizer of 1, and set $Z' = Z^{S_{m-1}}$. We also set $F'_m = Z^{S_m}$. We can choose a set of left coset representatives σ_k of S_{m-1} in S_m so that σ_1 is the identity and $\sigma_k(1) = k$. Then $Z_1 \subset Z'$ and we can set $UD'_1 = UD_1 \otimes_{Z_1} Z'$, $UD' = UD \otimes_{F(X,Y)} Z'$, and $L' = Z'(UD'_1 \otimes_{Z'} UD'^\circ)$.

Theorem 3.4. *F_m is isomorphic to the transfer over Z'/F'_m of the field extension L'/Z'.*

Proof. The transfer is the field of fractions of the invariant ring of

$$S = \bigotimes_{k=1}^{m} \sigma_k(Z \otimes_{Z'} L')$$

where $\sigma_k(Z \otimes_{Z'} L')$ refers to the σ_k twist and the iterated tensor product is over Z. To prove the theorem, it suffices to find an S_m invariant embedding $S \to L$ such that L is the field of fractions of the image of S. This map is actually obvious, being just $\phi(l_1 \otimes \ldots \otimes l_m) = \prod_k \sigma_k(l_k)$. That the field of fractions of $\phi(S)$ is L is clear, and that ϕ is injective can be seen by checking transcendence degrees. ∎

Next we analyze $Z/F(P,Q)$ a bit more. We begin with Z_1 which is the center of the generic division algebra $UD(F(P,Q),n,2)$. As such, Z_1 has the form $F(P,Q)(M_r)^{S_r}$ where M_r is the S_r lattice described in section one. Form the wreath product group $W_r = (S_r \oplus \ldots \oplus S_r) \rtimes S_m$ where there are m terms in the direct sum and the action of S_m is the obvious one. Let $A \subset W_r$ be the m fold direct sum $S_r \oplus \ldots \oplus S_r$ and $p'_1 : A \to S_r$ the projection on the first term. If we set $H = AS_{m-1} \subset W_r$, then p'_1 extends to $p_1 : H \to S_r$ by setting $p_1(S_{m-1}) = 1$. Using p_1 we can view M_r as an H module and set $N = \mathrm{Ind}_H^G(M_r)$. We claim:

Theorem 3.5. *$Z \cong F(P,Q)(N)^A$ with an isomorphism preserving the S_m actions. Thus $F'_m \cong F(P,Q)(N)^{W_r}$.*

Proof. Write $Z'_1 = F(P,Q)(M_r)$ so $Z_1'^{S_r} = Z_1$. Thus, as fields (i.e. ignoring group actions), $Z'' = Z'_1 \dots Z'_m = F(P,Q)(M_r \oplus \dots \oplus M_r)$ and this later field has an obvious action by A so that $Z = Z_1 \dots Z_m = F(P,Q)(M_r \oplus \dots \oplus M_r)^A$. Checking the S_m action makes it clear that Z'' has an induced W_r action and that with respect to this action $Z'' = F(P,Q)(N)$. The theorem is now clear. ∎

If we set $r = 2$ the above picture will begin to look very familiar. W_2 is the Weyl group W, and M_2 is the lattice $M \oplus \mathbb{Z}x_1 \oplus \mathbb{Z}x_2$, where M is as in the previous section. $N = \mathrm{Ind}_H^W(M \oplus \mathbb{Z}x_1 \oplus \mathbb{Z}x_2) = Y_D \oplus X$ where $X = \mathbb{Z}x_1 \oplus \dots \oplus \mathbb{Z}_{2m}$.

We form the generic 2×2 generic division division algebra over $F(Y_D)^W$ and write its center, following the above conventions, as $F(Y_D)^W(P,Q)$. We claim:

Lemma 3.6. *$F(Y_D)^W(P,Q)(y_1, \dots, y_{2m}) = F(P,Q)(N)^W$ is the field F'_m in the case $r = 2$.*

Proof. This follows immediately because $F(P,Q)(N) = F(P,Q)(Y_D \oplus X)$ and X is a permutation lattice. ∎

Next we look at the quaternion algebra $\Delta(F(Y_D)^{H'}/F(Y_D)^H, y_{12}y_{21})$ from the previous section. Tracing through the definitions, we find that:

Lemma 3.7. *$\Delta(F(Y_D)^{H'}/F(Y_D)^H, y_{12}y_{21}) \otimes_{F(Y)^H} Z' = UD'_1$ in the notation defined before 3.4.*

We now turn to the promised construction of a generic way to make a central simple algebra be in the image of restriction. Let A/F be a central simple algebra of degree r and suppose K/F is a separable degree m field extension. Assume B/K is also central simple of degree r. Form the generic splitting field $K(B \otimes_K A^\circ)$ and define $F(A,B)$ to be the transfer $\mathrm{Tr}_{K/F}(K((B \otimes_K A^\circ))$ of this field to F. We set $KF(A,B)$ to be $K \otimes_F F(A,B)$.

Proposition 3.8. *a) $B \otimes_K KF(A,B) \cong A \otimes_F KF(A,B)$.*

b) Suppose $B \cong B' \otimes_F K$ and $(r,m) = 1$. Then $F(A,B)(x_1, \dots, x_{r^2-1})$ is purely transcendental over $F(A \otimes_F B'^\circ)$.

c) $F(A,B)/F$ has transcendence degree $m(r^2 - 1)$.

Proof. $K((A \otimes_F K) \otimes_K B^\circ)$ is a subfield of $KF(A,B)$ and so $A \otimes_F KF(A,B)$ and $B \otimes_K KF(A,B)$ define equal elements in the Brauer group. Having equal degrees, they are isomorphic. This proves a). As for b), form $L = F(A,B)(A \otimes_F B'^\circ)$. Since A and B are equal in the Brauer group of $KF(A,B)$, taking corestrictions we have that A^m and B'^m are equal in the Brauer group of $F(A,B)$. Since $(r,m) = 1$, we have that A and B' are equal in the Brauer group of $F(A,B)$. Thus $L = F(A,B)(x_1, \dots, x_{r^2-1})$. We can also write $L = F(A \otimes B'^\circ)(A,B)$. For convenience, set $F' = F(A \otimes B'^\circ)$. Then $A \otimes_F F'K \cong B \otimes_K F'K$, and so $F'(A,B)$ is the transfer of the rational field extension $F'K((A \otimes_F F'K) \otimes_{F'K} (B \otimes_K F'K)^\circ)$ and thus is rational over F'. This proves b). The calculation of the transcendence degree is immediate. ∎

The way to use the $F(A,B)$ construction to **generically** force B to be in the image of restriction is to make A generic. That is, suppose K/F has degree m as above and B/K is central simple of degree r. Let $Z = Z(F,r,2)$ be the center of the generic division algebra $UD = UD(F,r,2)$. Set $B_Z = B \otimes_K KZ$ and then set $F_R(B) = Z(UD, B_Z)$. That is, $F_R(B)$ is the field defined by generically forcing B to be the image of UD/Z. Set $KF_R(B) = K \otimes_F F_R(B)$ and $UD_R = UD \otimes_Z F_R(B)$. We use 3.8 to show that:

Theorem 3.9. *$B \otimes_K KF_R(B) \cong UD_R \otimes_{F_R(B)} KF_R(B)$. If $B = B' \otimes_F K$, then $F_R(B)(x_1, \ldots, x_{r^2-1})$ is rational over $Z(UD \otimes B'^{\circ})$ of degree $m(r^2-1)$ which is in turn rational over F of transcendence degree $2r^2$. $F_R(B)$ has transcendence degree $m(r^2-1) + r^2 + 1$ over F.*

Proof. The first two statements follow from 3.8 a) and b). By 3.1 $Z(UD \otimes B'^{\circ})$ is rational over F of transcendence degree $2r^2$. The last statement follows by arithmetic. ∎

We apply this $F_R(B)$ construction some algebras that arose in section 2. For this reason, we again fix $r = 2$. Set $B = \Delta(F(Y_D)^{H'}/F(Y_D)^H, y_{12}y_{21})$ and $L' = F(Y_D)^H(y_1, \ldots, y_{2m})$. Set $L = F(Y_D)^W(y_1, \ldots, y_{2m})$. Part a) below will be proved by comparing the definition of $L_R(B)$ and 3.4.

Theorem 3.10.

a) $L_R(B) = (F(Y_D)^W(y_1, \ldots, y_{2m}))_R(B) = F_m = F(P, Q, P_1, \ldots, P_m, Q_1, \ldots, Q_m)^{S_m}$.

b) *$L_R(B)$ is rational over $F(P,Q)$ and F.*

c) *$(F(Y)^W)_R(B)(x_1, x_2, x_3)$ is rational over $Z(F(Y)^W, 2, 2)$.*

Proof. We begin with a). Theorem 3.4 describes F_m as a transfer, and $L_R(B)$ is defined as a transfer. The proof of a) then amounts to the verification that they are transfers of the same fields up to isomorphism. To prove b), all we need to remark is that 3.3 and the fact $F(P,Q)/F$ is rational finishes the argument. This later fact, that the center of generic 2×2 generic matrices is rational over F, is due to Procesi ([P1])]. Part c) is direct from 3.9. ∎

Section four: Finishing the proof

We are ready to prove the major theorem 4.2. We first dispose of the $n = 2$ case.

Lemma 4.1. *$F(V)^{PO_2}$ and $F(V)^{PSp_2}$ are both rational over F.*

Proof. Of course, $Z(F,2)$ is rational over F. If A is Brauer equivalent to $UD(F,2) \otimes UD(F,2)$, then A is split. Thus we are done by 1.1. ∎

Theorem 4.2. *Suppose $n = 2m$, m is odd. Let $V = M_n \oplus M_n$ be the representation of PSp_n and PO_n given by conjugation on each component. Then $F(V)^{PSp_n}$ and $F(V)^{PO_n}$ are rational over F.*

Proof. As we said at the beginning, it suffices to consider PSp_n and thus all the machinery of sections one through three. We can also assume $m \geq 3$. Form $(F(Y)^W)_R(B)$

and note that this can be written as $((F(Y_D))_R(B))_\beta(Y_0)^W$ where β denotes the extension $0 \to Y_D \to Y \to Y_O \to 0$. Since B, suitably extended, is in the image of restriction, we have that $((F(Y_D))_R(B))_\beta(Y_0) = ((F(Y_D))_R(B))(Y_0)$ with non-twisted W action. The extension $0 \to Y_O \to Y'_O \to I_m \to 0$ is easily seen to be defined by the image, γ_O, of the canonical $\gamma \in H^2(W, Y)$. But γ_O is trivial when restricted to H, so must have order dividing m. If we write $\gamma = \gamma_2 + \gamma_m$, where γ_i has order i, then γ_O must also be the image of γ_m. Let $B_m/F(Y)^W$ be the central simple algebra associated to γ_m of degree m. Then $(F(Y)^W(B_m))_R(B) = ((F(Y)^W)_R(B))(B_m) = (((F(Y_D))_R(B))(Y_0)^W)(B_m) = ((F(Y_D))_R(B))(Y'_0)^W$. Of course $2\gamma = 2\gamma_m$ and so we can set B'_m to be the degree m algebra associated with 2γ and derive that by Tregub's result (0.3 b))

$$((F(Y_D))_R(B))(Y'_0)^W = ((F(Y)^W)_R(B))(B_m) = \\ ((F(Y)^W)_R(B))(B'_m) = ((F(Y)^W)(B'_m))_R(B). \tag{2}$$

We will analyze further both ends of (2). Beginning on the left, Y'_O is a permutation W lattice and so by 0.2 $((F(Y_D))_R(B))(Y'_0)^W$ is rational over $(F(Y_D)^W)_R(B)$ of degree the rank of Y'_O which is $4m(m-1)$. Since $4m(m-1) \geq 2m$ for $m \geq 3$, this last field is rational over F by 3.10 b).

Turning to the right end of (2), $B \otimes_{F(Y_D)^H} F(Y)^H$ is in the image of restriction, namely it is the image of the quaternion algebra over $F(Y)^W$ associated to γ_2. Thus, by 3.9, $(((F(Y)^W)(B'_m))_R(B))(x_1, x_2, , x_3)$ is rational over $((F(Y)^W)(B'_m)$ of degree $3m+8$. We showed in 1.7 that $F(V)^{PSp_n}$ is rational over $F(Y)^W(B'_m)$ of degree $2m^2$. But $2m^2 > 3m+8$ if $m \geq 3$. This proves 4.2. ∎

Section Five: Four times an odd

In this section we will assume $n = 4m$, n is odd, and show

Theorem 5.1. *$F(V)^{PSp_n}$ and $F(V)^{PO_n}$ are stably rational.*

As before, it suffices to consider the former case.

Now when it comes to stable rationality we can quickly simplify our situation. In [K], [Sc] it was shown that $Z(F,n)$ was stable isomorphic to $Z(F,4)Z(F,m)$. We gave a later proof of this in [S1]. Let L be the field from [S1] rational over both $Z(F,n)$ and $Z(F,4)Z(F,m)$. In [S1] we also showed that $UD(F,n) \otimes_{Z(F,n)} L \cong (UD(F,4) \otimes_{Z(F,4)} L) \otimes_L (UD(F,m) \otimes_{Z(F,m)} L)$. That is, $UD(F,n)/Z(F,n)$ is stably isomorphic to $UD(F,4)$ $\otimes UD(F,m)$.

Let B be Brauer equivalent to $UD(F,n) \otimes UD(F,n)$. Write $B = B_2 \otimes B_m$ where B_2 has 2 power degree and B_m has odd degree. By e.g. [S1] p. 392, $L(B)$ is stably isomorphic to $L(B_2)(B_m^\circ)$ which is in turn obviously stably isomorphic to the join $(Z(F,4)(B_2)(Z(F,m)(B_m))$. Now B_m is Brauer equivalent to $UD(F,m) \otimes UD(F,m)$ so $Z(F,m)(B_m)$ is stably isomorphic to $Z(F,m)(UD(F,m))$ (we do not need Tregub's stronger result here). Of course, this last field is rational over F. Thus we have shown:

Lemma 5.2. *To prove 5.1 it suffices to show $F(V)^{PSp_4}$ is stably rational over F.*

From now on, then, we only deal with the $n = 4$ case. We know $F(V)^{PSp_4}$ is rational over $F(Y_2)^W$. In this case $W = A \rtimes \mathbb{Z}/2\mathbb{Z}$ has order 8. Let σ generate the $\mathbb{Z}/2\mathbb{Z}$ part and $A =< \sigma_1 > \oplus < \sigma_2 >$ such that $\sigma(\sigma_i)\sigma = \sigma_{3-i}$. Let C be the subgroup generated by $\sigma\sigma_1$ of order 4. In particular, W can also be described as a dihedral group of order 8. Note that $H' = W \cap S_3$ is just $< \sigma_2 >$ and $H = A \rtimes S_{m-1}$ is just A in this case. Finally let B be the subgroup of order 4 generated by $\sigma_1\sigma_2$ and σ.

Set the lattice $Y' = Y'_D \oplus Y'_O$ just as above, and note that the restriction of $Y' \to I[W/H']$ to Y'_O is onto. Also note that $Y'_O \cong \mathbb{Z}[W]$ (generated by y_{31}). We will be using 0.2 several times in the rest of this argument.

If Y_4 is the kernel of $Y'_O \to I[W/H']$, then we have $0 \to Y_4 \to Y \to Y'_D \to 0$. By 0.2, it suffices to examine $F(Y_4)^W$. Note that Y_4 has rank 5. The canonical generator of $\mathbb{Z}[W]$ maps to $d_3 - d_1$. It follows that $1 + \sigma \in Y_4$. Thus $\mathbb{Z}[G/ < \sigma >] \subset Y_4$. We need to find one more element in Y_4 and it is clear that it is $(1 - \sigma_1)(1 - \sigma_2)$.

Of course what we really have to explore is the lattice Y_2. Recalling 1.3 we can take for Y_2 (here we are using 0.2 again) the preimage in $\mathbb{Z}[W]$ of $d_1 + d_2 - d_3 - d_4$. That is, Y_2 is generated by Y_4 and $(\sigma_1 + \sigma_2)$. All together, Y_2 is generated by $\mathbb{Z}[W/ < \sigma >](1 + \sigma)$, $1 + \sigma_1\sigma_2$ and $\sigma_1 + \sigma_2$.

Let B be as above, and $M \subset Y_2$ the B module generated by $\mathbb{Z}[B/ < \sigma >](1 + \sigma)$ and $1 + \sigma_1\sigma_2$. Then it is easy to see that $Y_2 \cong \mathrm{Ind}_B^W(M)$. It helps a bit to work with M. To simplify, set $B =< a > \oplus < b >$ where $b = \sigma$ and $a = \sigma_1\sigma_2$. Let $x \in M$ be $1 + b$ and $y = 1 + a$. Then M is generated by x, y subject to the relations that b fixes x, a fixes y and $(1 + a)x = (1 + b)y$. Note that this last relation can be equivalently written as $(1 + ab)(x - y) = 0$.

We define an embedding $M \to \mathbb{Z}[B] \oplus \mathbb{Z}$ by sending x to $(1+b, 1)$ and y to $(b+ab, 1)$. Computing the cokernel we have the exact sequence $0 \to M \to \mathbb{Z}[B] \oplus \mathbb{Z} \to \mathbb{Z}[B/ < ab >] \to 0$. Inducing up to W we have $0 \to Y_2 \to \mathbb{Z}[W] \oplus \mathbb{Z}[W/B] \to \mathbb{Z}[W/ < \sigma_1\sigma_2\sigma >] \to 0$. We have almost shown:

Lemma 5.4. *To prove 5.1 it suffices to show $F(U)^W/W$ is rational for some faithful W representation U.*

Proof. The above argument reduces us to considering $F(Z[W] \oplus \mathbb{Z}[W/B])^W$ which can also be written $F(U')^W$ for the corresponding permutation representation $U' = F[W] \oplus F[W/B]$. By Endo-Miyata's result (0.2), stably rationality for one faithful representation implies it for all. ∎

Let U be the dual of the two dimensional representation $Fv_1 + Fv_2$ where $\sigma_i(v_i) = -v_i$ $\sigma_i(v_j) = v_j$ if $i \neq j$, and $\sigma(v_i) = v_{3-i}$. Then we can view $F(U) = F(v_1, v_2)$ and it is clear that $F(U)^W = F(v_1^2 + v_2^2, v_1^2 v_2^2)$ (Note also that W is a reflection group on U so we could quote Chevalley). Theorem 5.1. is proven.

Appendix

The initial version of this paper we assumed the ground field F was algebraically closed of characteristic 0. In this appendix we gather the arguments that allow us

to assume F is any field. There are two key results in this section. The first, A.1, shows that the standard section argument applies in any characteristic, avoiding use of Zariski's Main Lemma. The second result, in A.3, avoids use of the proof of Bogomolov's No name Lemma. We also note that this whole section only deals with the group PSp_n but it is clear that other, less elementary methods, could be employed for arbitrary reductive groups.

If V is a representation of PSp_n, we say V is **good** if there is an affine open subset $U_1 \subset V$ such that all points of U_1 have trivial stabilizer. Let $M^- \subset M_n(F)$ be the space of skew symmetric matrices and $\Delta \subset M^-$ the subspace of diagonal matrices. Set $V^- = M^- \oplus V$ and $V^\Delta = \Delta \oplus V$. If $x \in V^-$ or $y \in V^\Delta$ we set $\bar{x} \in M^-$ or $\bar{y} \in \Delta$ to be the projection on the first summand. Let $T_{PSp} \subset N_{PSp} \subset PSp_n$ be the maximal torus and its normalizer described in section one. Set $W = N_{PSp}/T_{PSp}$ to be the Weyl group.

Theorem A.1. *Let V be a good representation and let M^-, Δ, V^-, V^Δ and N_{PSp} be as above. Then $F(M^- \oplus V)^{PSp_n} = F(\Delta \oplus V)^{N_{PSp}}$.*

Proof. The action $PSp_n \times V^- \to V^-$ is the obvious one. We consider the variety $PSp_n \times (V^\Delta)$. This variety has the N_{PSp} action $n \cdot (g, x) = (gn^{-1}, nx)$ and we will write the quotient as $PSp_n \times_N (V^\Delta)$. Since $gx = (gn)(n^{-1}x)$, the PSp_n action induces $\phi : PSp_n \times_N (V^\Delta) \to (V^-)$.

Lemma A.2. *ϕ restricts to an open immersion on some open subset of $PSp_n \times_N (V^\Delta)$.*

Proof. Let $X \subset PSp_n \times V^- \times V^\Delta$ be the closed reduced subvariety of (g, x, y) such that $gy = x$. Clearly the projection $X \to PSp_n \times V^\Delta$ is an isomorphism. X has an action by N via $n \cdot (g, x, y) = (gn^{-1}, x, ny)$ and a commuting action by PSp_n given by $g' \cdot (g, x, y) = (g'g, g'x, y)$. The N action on X translates to the N action on $PSp_n \times V^\Delta$. The "action" map $PSp_n \times V^\Delta \to V^-$ translates to the projection $\pi : X \to V^-$. Since the generic element of M^- is diagonalizable, π is dominant. Of course, π induces $X/N \to V^-$.

This last projection can be factored into $X \to V^- \times \Delta \to V^-$ where the first map, π_1, is $(g, x, y) \to (x, \bar{y})$ and the second map, π_2, is the obvious projection. Note that $T_{PSp} \subset N_{PSp}$ acts trivially on $V^- \times \Delta$ so that π_1 factors through X/T. Let Y be the closure of the image of π_1. Note that W acts on Y but as W acts trivially on V^- we have an induced $Y/W \to V^-$.

The image of π_1 contains the dense subvariety defined by $p_{\bar{x}}(t) = \prod_i (t - \theta_i)$ and $d_{\bar{x}} \neq 0$ where $p_{\bar{x}}(t)$ is the characteristic polynomial of $\bar{x}$, $d_{\bar{x}}$ is the discriminant of $p_{\bar{x}}(t)$, and the θ_i are the (diagonal) entries of $\bar{y}$. Thus the extension $F(V^-) \subset F(Y)$ amounts to adjoining roots of the generic $p_x(t)$ and so $F(Y)^W = F(V^-)$. This implies there is an open subset U_Y of Y with $U_Y/W \to V^-$ an open immersion.

Thus the lemma will be proven if we show $X/T_{PSp} \to Y$ restricts to an open embedding. To show this, we will show that there is a rational section $s : Y \to X$. If $(x, \bar{y}) \in Y$, let θ_i be the entries of $\bar{y}$. Then $\theta_i 1 - x$ is singular, and so $(\theta_i 1 - x))(\theta_i 1 - x)^* = 0$ where z^* is the adjoint of z. Let v_i be the first column of $(\theta_i 1 - x)^*$. For some $U'_Y \subset Y$ open, all θ_i are distinct and all the $v_i \neq 0$. The v_i are, of course, eigenvectors for x with eigenvalue θ_i. By the definition of the skew involution J, $\theta_{2i-1} + \theta_{2i} = 0$. Let (v, v')

be the skew form associated with J. That is, if e_i is the standard basis, $(e_i, e_j) = 0$ if $|i-j| \neq 1$ and $(e_{2i-1}, e_{2i}) = 1 = -(e_{21}, e_{2i-1})$. Now it is immediate that $(v_i, v_j) = 0$ if $\theta_i + \theta_j \neq 0$. Since the form is nondegenerate, it is also immediate that $(v_{2i-1}, v_{2i}) \neq 0$. We define $v'_{2i-1} = v_{2i-1}$ and $v'_{2i} = (1/(v_{2i-1}, v_{2i}))v_{2i}$. If $a'(x, \bar{y})$ is the matrix with i column v'_i, then $a'(x, \bar{y}) \in Sp_n$ and $a'(x, \bar{y})\bar{x}a'(x, \bar{y})^{-1} = \bar{y}$. Thus if $a(x, \bar{y})$ is the image of $a'(x, \bar{y})$ in PSp_n, the map $(x, \bar{y}) \to (a(x, \bar{y}), x, \bar{y} + a'(x, \bar{y})(x - \bar{x}))$ is a section $U'_Y \to X$ of $X \to Y$. The composition $U'_Y \to X \to X/T_{PSp}$ is a rational inverse to the map $X/T \to Y$. All together, Lemma A.2 is proven.

We now turn to the proof of Theorem A.1. Let $U' \subset M^-$ be the affine open subset of elements $\bar{x}$ with $d_{\bar{x}} \neq 0$. Then U' is a union of PSp_n orbits, all of which are closed. Furthermore, any two elements of $U' \cap \Delta$ are in the same N_{PSp} orbit. Let $p : V^{-1} \to M^-$ be the projection. There is an affine open $U'' \subset V^-$ such that all points of U'' have trivial stabilizer. For example, we can choose $U'' = M^- \times U_1$.

Let $U \subset V^-$ be $p^{-1}(U') \cap U''$. Set $U_\Delta = V^\Delta \cap U$. Then restriction defines a morphism $F[U] \to F[U_\Delta]$ which obviously induces $\phi : F[U]^{PSp_n} \to F[U_\Delta]^{N_{PSp_n}}$. If $f \in F[U]^{PSp_n}$ satisfies $\phi(f) = 0$, then f is 0 on U_Δ. But all the PSp_n orbits of U meet U_Δ, and f is constant on such orbits, so $f = 0$. Thus ϕ is an injection. By 1.4, $F(V^-)^{PSp_n}$ is the field of fractions of $F[V^-]^{PSp_n}$ and hence of $F[U]^{PSp_n}$. Thus ϕ induces an embedding $\phi' : F(V^-)^{PSp_n} \to F(V^\Delta)^{N_{PSp}}$.

As a T_{PSp_n} module, V is a direct sum of spaces of the form Fv_τ where $\eta(v_\tau) = \tau(\eta)v_\tau$ and τ is a character of T (e.g. [K-T] p. 343). Since V is good, it follows that the τ that appear generate the full group of characters of T. Since N_{PSp} also acts on T, it follows that if τ appears in V then so does τ^{-1}. Thus 1.4 applies to V^Δ and $F(V^\Delta)^{N_{PSp}}$ is the field of fractions of $F[V^\Delta]^{N_{PSp}}$.

We can use Lemma A.2 to show ϕ is birational. It suffices to show that if $f \in F[V^\Delta]^{N_{PSp}}$, then f is in the image of ϕ'. Let $f' \in F[PSp_n \times V^\Delta]$ be defined by $f'(g, v) = f(v)$. Since f is N invariant, $f' \in F[PSp_n \times_N V^\Delta]$ and so by A.2 f' defines a rational function f'' on V^- and the definition of f and f' show that f'' is PSp_n invariant. It is clear that f is the image of f''. This proves A.1, the first of the two results we needed in this section. ∎

The second result we need is the rationality of $F(M_n(F) \oplus M_n(F))^{PSp_n}/F(M^- \oplus M_n(F))^{PSp_n}$. When F is algebraically closed of characteristic 0, this is an immediate consequence of the argument in Bogomolov's "No-Name" lemma. The point is that all PSp_n representations are completely reducible. In order to handle general F, we use A.1. to reduce to the case of finite groups, and then we use the result of Endo-Miyata (0.2). To be precise, we will prove the following.

Theorem A.3. *Suppose V is a good representation as in A.1 and M^-, V^- are also as described there. Let $V' = V \oplus V''$ where V'' is any PSp_n representation. Write $V'^- = M^- \oplus V'$. Then $F(V'^-)^{PSp_n}/F(V^-)^{PSp_n}$ is rational.*

Proof. First of all, it is clear that V' is also a good representation. Thus the theorem reduces to showing $F(V'^\Delta)^{N_{PSp}}/F(V^\Delta)^{N_{PSp}}$ is rational, where V^Δ is as above and of course $V'^\Delta = \Delta \oplus V'$. We observed above that for each $\tau \in \mathrm{Hom}(T, F^*)$ there was a monomial $m_\tau \in F[V]$ such that $\eta(m_\tau) = \tau(\eta)^{-1}m_\tau$. V'' has a basis of v_i such that

$\eta(v_i) = \tau_i(\eta)v_i$ and we set $m_i = m_{\tau_i}$. Let V_1 be the F span of the $m_i v_i$. It is clear that $F(V'^{\Delta}) = F(V^{\Delta} \oplus V_1)$, T acts trivially on V_1, and that $F(V'^{\Delta})^T = F(V^{\Delta})^T(V_1)$. Now V_1 is not preserved by W so we set $V_2 = F(V^{\Delta})^T V_1$.

We claim V_2 is preserved by the action of W. For any $\tau \in \mathrm{Hom}(T, F^*)$, set $V''_\tau = \{v \in V'' | \eta(v) = \tau(\eta)v \text{ all } \eta \in T\}$. Since for each τ we chose a unique m_τ, $m_\tau V''_\tau$ is a subspace of V_1. It is clear that for $\sigma \in N_{PSp_n}$, $\sigma(V''_\tau) = V''_{\sigma(\tau)}$. Then

$$\sigma(m_\tau V''_\tau) = (\sigma(m_\tau)/m_{\sigma(\tau)})m_{\sigma(\tau)}V_{\sigma(\tau)}.$$

But $(\sigma(m_\tau)/m_{\sigma(\tau)}) \in F(V^{\Delta})^T$ and so it is clear N (i.e. W) preserves V_2.

Of course $F(V^{\Delta})^T(V_1) = F(V^{\Delta})^T(V_2)$. W acts faithfully on $F(V^{\Delta})^T$ and so by Endo-Miyata (0.2) $F(V^{\Delta})^T(V_2)^W$ is rational over $(F(V^{\Delta})^T)^W$. But $F(V^{\Delta})^T(V_2)^W = F(V'^{\Delta})^N$ and $(F(V^{\Delta})^T)^W = F(V^{\Delta})^N$. ∎

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